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An investigation of the feasibility of reducing the size of linear programming problems

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CONTENTS

Abstract

This investigation has been conducted to explore the various techniques which can eventually reduce the size of linear programming problems by reducing the number of variables and constraints.

A technique has been suggested to handle the equality type constraints which will reduce the number of variables. Further, an attempt has been made to extend this technique to inequality type of constraints.

Also, some techniques have been brought into light which can detect infeasibility and unboundness of the problem. They can further detect the extraneous variables which will not appear in the final tableau and will provide information about the redundant constraints.

Other techniques have also been included which were fully investigated. These can be applied only to particular cases of some problems, but they certainly reduce the size of a problem under attack with respect to number of variables. Finally, a flow chart is presented as a guide in applying the techniques discussed in this thesis.

Chapter I

Introduction

The human mind demands unity of method and doctrine. The construction of a scientific framework requires a certain respect for the facts of experience, open-mindedness, an experimental trial and error attitude and the capacity for working within an incomplete framework. Einstein, in attempt to differentiate among the various mental attitudes which may be found in a scientist, points out that he may appear as a realist, an idealist or a platonist. As a realist, he attempts to describe objectively the facts of experience. As an idealist, he depends on his theories and ideas as means for constructing convenient concepts. As a platonist, he considers logical simplicity as an essential tool for his research.

A frequent problem which usually occurs in the operations research is the optimization of resources, profit, cost, etc. Linear programming is the simplest and strongest tool of operations research and is concerned with the problem of optimizing a linear portion of several variables subject to linear constraints.[1]

Linear programming deals with the constraints and function to be optimized. The constraints may be algebraic equations or inequalities of arbitrary degree, the variables always being constrained to be positive or zero. The method of linear programming is endowed with a special characteristic common to other techniques of applied mathematics: a compelling need for efficiency. Knowing how to solve a problem in a finite number of operations is not enough, the amount and cost of the time needed to obtain a soluition must be within reasonable limits.

I will concentrate on the aspects which can lead to simplification of linear programming problems as Einstein points out that simplification is an essential tool for research. My objective throughout the thesis will be stressed upon the facts which can lead to the simplification and thus reduction of the size of linear programming problems.

In this area, work has been done by many authors. Mr. Fettor has suggested a method with which all constraints are examined at the end of each simplex iteration and some of them removed by a method established by him, without affecting the optimality conditions of a problem. (Speed Up Solution To LP Problems, Journal of Industrial Engineering, 1961)

Mr. Jacques De Buchet (France) deals with choice of algorithm and inversion of matrix, and development of operating system without human intervention. (Solution of Large Scale LP Problems, 4070 International Abstracts in Operations Research, 1966)

Mr. Leon S. Lasdon discussed some algorithms for optimizing large systems. But, these algorithms can be applied to problems only with some specific structure. One of them discussed in his book is "Dantzig-Wolfe Decomposition Principle." (Optimization Theory For Large Systems, MacMillan Co., 1970) Mr. Gass[2] discusses a method which can be applied when all the constraints are of greater than or equal to type and further discusses a method which there is a mixture of inequalities.

Mr. Robert W. Llewellyn[3] has given a full chapter in his book "Algebraic Simplification and Allied Topics" in which he explores many areas for simplification. Some of them, I have included in my thesis, too.

The basic problem in linear programming is merely to maximize (or minimize) a linear function of the form,

$$X_o = C_1 X_1 + C_2 X_2 + \text{-----} + C_n X_n$$

where C_1, C_2, --- C_n are known. X_1, X_2, --- X_n have to be found out. If we assume that variables can assume any variables, the solution is trivial. But, actually this function is subjected to many restrictions. These can be stated mathematically as,

$$\left.\begin{array}{l} a_{11} X_1 + a_{12} X_2 + \text{-----} + a_{1n} X_n \leq b_1 \\ a_{21} X_1 + a_{22} X_2 + \text{-----} + a_{2n} X_n \leq b_2 \end{array}\right. \quad (1.1)$$

$$\left.\begin{array}{l} a_{k1} X_1 + a_{k2} X_2 + \text{-----} + a_{kn} X_n = b_k \\ a_{k+1,1} X_1 + a_{k+1,2} X_2 + \text{-----} + a_{k+1,n} X_n = b_{k+1} \end{array}\right. \quad (1.2)$$

$$a_{m-1,1}\ X_1 + a_{m-1,2}\ X_2 + \text{-----} + a_{m-1,n}\ X_n \geq b_{m-1}$$

$$a_{m1}\ X_1 + a_{m2}\ X_2 + \text{-----} + a_{mn}\ X_n \geq b_m \qquad (1.3)$$

All the values of b_i's are assumed positive.

Values of variables are further restricted:

$$X_j \geq 0 \qquad j = 1,2, \text{--}\ n \qquad (1.4)$$

Constraints of (1.1) form will be termed as Type I inequality type of constraint.

Constraints of (1.2) form will be termed as equality type of constraints.

Constraints of (1.3) form will be referred as Type II inequality type of constraints.

Constraints of (1.4) form are in general called non-negativity constraints on variables.[4]

To solve the problem subjected to the constraints as stated above is quite interesting. The simplex method invented by Dr. George B. Dantzig[5] can handle such problems.

To solve a linear programming problem by the simplex method, one has to write the problem in the standard form i.e. appropriate slack and artificial variables should be added and subtracted to convert the constraints into equations, such that we have an initial feasible basic solution to the problem. To handle Type I constraints, one has to add a slack variable; to handle equity type constraints, one has to add artificial variables; to handle Type II constraints, one has to subtract a slack and add artificial variables. It can be shown by an example.

<u>Ex. 1.5</u>

$$\text{Max.} \quad 2X_1 - 3X_2 + 4X_3$$

$$\text{S/T} \quad 3X_1 + X_2 - 2X_3 \leq 6 \qquad 1.51$$

$$X_1 + 2X_2 + X_3 = 7 \qquad 1.52$$

$$2X_1 - X_2 + X_3 \geq 9 \qquad 1.53$$

Can be written in the standard form.

$$\text{Max.} \quad 2X_1 - 3X_2 + 4X_3 + OS_1 - MR_2 + OS_3 - MR_3$$

$$\text{S/T} \quad 3X_1 + X_2 - 2X_3 + S_1 + OR_2 + OS_3 + OR_3 = 6$$

$$X_1 + 2X_2 + X_3 + OS_1 + R_2 + OS_3 + OR_3 = 7$$

$$2X_1 - X_2 + X_3 + OS_1 + OR_2 - S_3 + R_3 = 9$$

Initial feasible basic solution to the problem is, $S_1 = 6$, $R_2 = 7$, $R_3 = 9$.

Now the problem can be tabulated and solved by using the standard simplex method. This problem has seven variables and three constraints. So, it would have seven columns and three rows in the simplex tableaus.

But, at the same time, the question comes to mind, is it possible to reduce the size of problem with respect to number of constraints or variables so that size of tableau should be small? I think that the answer is yes, at least for this problem. How that can be done, will be dealt in the following chapters.

Chapter II

Equality Type of Constraint

A Linear Programming problem can have various types of constraints - Type I, Type II and equality. Similarly, it can have various types of restrictions on variables - non-negativity, non-positivity, unrestricted in sign, upper bound or lower bound. However, it is necessary to mention that the simplex method can be applied only if - non-negativity restriction prevails on variables. If the former is not the case, then necessary modification must be made so that simplex method can be implemented.

First, I will deal with the equality type of constraints and investigate the possibility of reducing the size of the problem. For simplicity, a two variable problem will be considered.

Ex. 2.1

Max. $X_0 = C_1X_1 + C_2X_2$

$$\text{S/T} \quad a_{11}X_1 + a_{12}X_2 \leq b_1 \qquad (2.11)$$

$$a_{21}X_1 + a_{22}X_2 = b_2 \qquad (2.12)$$

$$X_1, X_2 \geq 0$$

If we write it in the standard form, it will have four variables and one of these will be artificial.

But if transformation is performed from the constraint 2.12, one gets

$$X_1 = \frac{b_2}{a_{21}} - \frac{a_{22}}{a_{21}} X_2 \qquad (2.121)$$

substituting it into 2.11,

$$X_2 \leq \frac{b_1 a_{21} - a_{11} b_{21}}{a_{12} a_{21} - a_{11} a_{22}} \qquad (2.111)$$

Also, additional constraint

$$\frac{a_{22}}{a_{21}} X_2 \leq \frac{b_2}{a_{21}} \qquad (2.122)$$

must be added to ensure non-negativity on X_1. The modified problem is,

$$\text{Max.} \quad C_1 \frac{b_2}{a_{21}} + (C_2 - C_1 \frac{a_{22}}{a_{21}}) X_2$$

$$\text{S/T} \quad X_2 \leq \frac{b_1 a_{21} - a_{11} b_{21}}{a_{12} a_{21} - a_{11} a_{22}} \qquad (2.112)$$

$$X_2 \leq \frac{b_2}{a_{22}} \qquad (2.123)$$

Now, the problem has only one basic variable, so problem can just be solved by inspection and we do not have to go to simplex method.

It can be illustrated by taking examples.

<u>Ex. 2.2</u>

$$\text{Max.} \quad X_o = 10 X_1 + 5X_2$$

$$\text{S/T} \quad 2X_1 + X_2 \leq 5 \qquad (2.21)$$

$$X_1 + 2X_2 \leq 12 \qquad (2.22)$$

$$3X_1 + X_2 = 5 \qquad (2.23)$$

$$X_1 + X_2 \geq 2 \qquad (2.24)$$

$$10\ X_1 + 3X_2 \geq 5 \qquad (2.25)$$

$$X_1,\ X_2 \geq 0$$

From constraint 2.13,

$$X_2 = 5 - 3X_1 \qquad (2.231)$$

After substitution,

Max. $\quad X_o = 25 - 5X_1$

S/T $\quad X_1 \geq 0 \qquad (2.211)$

$$X_1 \geq -\frac{2}{5} \qquad (2.221)$$

$$X_1 \leq \frac{5}{3} \qquad (2.232)$$

$$X_1 \leq \frac{3}{2} \qquad (2.241)$$

$$X_1 \geq -10 \qquad (2.251)$$

$$X_1 \geq 0$$

Note that the constraint 2.232 has been added to ensure non-negativity on X_2.

The problem is simplified into one variable maximization problem. So, it can be solved just by inspection.

The optimal solution is,

$X_o = 25 \qquad X_1 = 0$

$X_2 = 5 - 3X_1 = 5$

When solving the problem by the simplex method, 9 variables and 5 constraints were required. The optimal solution was obtained after 5 iterations.

It is interesting to note that if the problem is changed from that of maximization to one of minimization, the solution can still be obtained by inspection, and no additional work is required. This is not the case with the simplex method.

Optimal solution is,

$$X_o = 17.5$$

$$X_1 = \frac{3}{2}$$

$$X_2 = 5 - 3 \times \frac{3}{2} = \frac{1}{2}$$

When the problem was solved by the simplex method, it required 4 iterations to reach the optimal solution.

Ex. 2.3 If one variable has a negative coefficient and one, a positive coefficient in the equality type of constraint, in that case, the variable with the positive coefficient should be eliminated.

e.g. Max. $X_o = 5X_1 + X_2$

S/T $2X_1 + X_2 \leq 6$ (2.31)

$-X_1 + 3X_2 = 5$ (2.32)

$X_1, X_2 \geq 0$

From 2.32,

$$X_2 = \frac{5}{3} + \frac{1}{3} X_1 = \frac{5}{3} - \left(- \frac{1}{3} X_1 \right)$$

Thus modified problem is,

$$\text{Max.} \quad X_o = \frac{5}{3} + \frac{16}{3} X_1$$

$$\text{S/T} \quad X_1 \leq \frac{13}{7} \qquad (2.311)$$

$$X_1 \geq - 5 \qquad (2.321)$$

The constraint 2.321 has been necessitated to ensure non-negativity on X_2. Solution by inspection is,

$$X_1 = \frac{13}{7} \qquad X_2 = \frac{5}{3} + \frac{1}{3} \times \frac{13}{7} = \frac{16}{7}$$

$$X_o = \frac{5}{3} + \frac{16}{3} \times \frac{13}{7} = \frac{81}{7}$$

Now, let me extend it to the problem which has a large number of variables. The method is the same, so, I will illustrate by taking examples.

Ex. 2.4

$$\text{Max.} \quad X_o = 2X_1 + 3X_2 + 5X_3$$

$$\text{S/T} \quad 2X_1 + 3X_2 + 4X_3 \leq 24 \qquad (2.41)$$

$$3X_1 + X_2 + 2X_3 \leq 18 \qquad (2.42)$$

$$5X_1 + 2X_2 + X_3 = 20 \qquad (2.43)$$

$$X_1, X_2, X_3 \geq 0$$

When dealing with more than two variables, the decision of which variable to eliminate requires some consideration, since proper choice of the variable which is to be eliminated may further simplify the problem.

If there are more Type I constraints, then the variable with the highest positive coefficient in the equality constraint should be removed. On the other hand, if more Type II constraints are present, then the variable with the least positive coefficient in the equality constraint is the variable which should be eliminated. This suggestion has been made in an attempt to have more Type I constraints and fewer Type II constraints as Type I are easy to handle.

For this problem, X_1 should be eliminated.

$$X_1 = 4 - \frac{2}{5} X_2 - \frac{1}{5} X_3$$

$$= 4 - (\frac{2}{5} X_2 + \frac{1}{5} X_3)$$

Modified problem is,

$$\text{Max.} \quad X_o = 8 + \frac{11}{5} X_2 + \frac{23}{5} X_3$$

$$\text{S/T} \quad 11X_2 + 18X_3 \leq 80 \qquad (2.411)$$

$$-X_2 + 7X_3 \leq 30 \qquad (2.421)$$

$$\frac{2}{5} X_2 + \frac{1}{5} X_3 \leq 4 \qquad (2.431)$$

$$X_2, X_3 \geq 0$$

The constraint 2.431 has been added to ensure non-negativity on X_1.

If it is written in the standard form, the problem would have five variables and three constraints, while the original problem had three constraints and six variables out of which one was artificial.

This modified problem was solved by the simplex method in two iterations.

The optimal solution is,

$$X_o = \frac{538}{19} \qquad X_2 = \frac{4}{19} \qquad X_3 = \frac{82}{19}$$

$$X_1 = 4 - \frac{2}{5} \times \frac{4}{19} - \frac{1}{5} \times \frac{82}{19} = \frac{58}{19}$$

While the original problem was solved in three iterations.

Ex. 2.5 Taking the example 1.5,

Max. $2X_1 - 3X_2 + 4X_3$

S/T $3X_1 + X_2 - 2X_3 \leq 6$ (2.51)

$X_1 + 2X_2 + X_3 = 7$ (2.52)

$2X_1 - X_2 + X_3 \geq 9$ (2.53)

$X_1, X_2, X_3 \geq 0$

Over here, one finds difficulty in making a choice of which variable to eliminate. Variable X_3 is arbitrarily chosen since its elimination will not involve a cumbersome transformation.

$$X_3 = 7 - X_1 - 2X_2 = 7 - (X_1 + 2X_2)$$

After substitution,

Max. $28 - 2X_1 - 11X_2$

S/T $X_1 + X_2 \leq 4$ (2.511)

$X_1 + 2X_2 \leq 7$ (2.521)

$X_1 - 3X_2 \geq 2$ (2.531)

$X_1, X_2 \geq 0$

Constraint 2.521 has been added to ensure non-negativity on X_3.

This problem was solved with one iteration and it had six variables and three constraints.

Optimal solution is,

$X_1 = 2$, $X_2 = 0$, $X_o = 24$, $X_3 = 7 - 2 = 5$

The original problem had seven variables and three constraints and it took two iterations to get optimal solution.

However, one must note that the number of iterations will not always be reduced by use of this technique. This would be true only if the variable that is eliminated also appears in the final tableau.

In case, we have more than one equality type of constraint, it becomes quite interesting to apply this technique.

Ex. 2.6

Max. $X_o = X_1 + 2X_2 + X_3$

$$\text{S/T} \quad 2X_1 - 2X_2 + X_3 = 3 \qquad (2.61)$$

$$3X_1 + X_2 - X_3 = 10 \qquad (2.62)$$

$$-X_1 + 5X_2 + 3X_3 \leq 12 \qquad (2.63)$$

$$X_1, X_2, X_3 \geq 0$$

Solve constraints 2.61 and 2.62 for any two variables, say X_1 and X_3, then we get,

$$X_1 = \frac{13}{5} + \frac{1}{5} X_2 = \frac{13}{5} - \left(- \frac{X_2}{5}\right)$$

$$X_3 = \frac{-11}{5} + \frac{8}{5} X_2 = \frac{-11}{5} - \left(-\frac{8}{5} X_2\right)$$

After substitution,

$$\text{Max.} \quad X_o = \frac{2}{5} + \frac{19}{5} X_2$$

$$\text{S/T} \quad \frac{-X_2}{5} \leq \frac{13}{5} \qquad (2.611)$$

$$-\frac{8}{5} X_2 \leq \frac{-11}{5} \qquad (2.621)$$

$$48X_2 \leq 106 \qquad (2.631)$$

Constraints 2.611 and 2.621 have been added to ensure non-negativity on X_1 and X_3. Can be rewritten as,

$$\text{Max.} \quad X_o = \frac{2}{5} + \frac{19}{5} X_2$$

$$\text{S/T} \quad -X_2 \leq 13 \qquad (2.612)$$

$$X_2 \geq \frac{11}{8} \qquad (2.622)$$

$$X_2 \leq \frac{53}{24} \qquad (2.632)$$

The problem has only one variable and so it can be solved simply by inspection.

Thus, $X_2 = \frac{53}{24}$

$$X_1 = \frac{13}{5} + \frac{1}{5} \times \frac{53}{24} = \frac{73}{24}$$

$$X_3 = -\frac{11}{5} + \frac{8}{5} \times \frac{53}{24} = \frac{4}{3}$$

$$X_o = \frac{2}{5} + \frac{19}{5} \times \frac{53}{24} = \frac{211}{24}$$

The same results were obtained by the simplex method after three iterations and the original problem had six variables and three constraints when set up for simplex.

Now, advantages of this technique can be visualized. Not only, one reduced the artificial variables, but also reduces the number of variables.

In a similar fashion, this method can be applied to any problem whether large or small and thus great deal of computational work can be avoided.

Applications

Dual Simplex Method: This method can be applied only if the condition $Z_i - C_i \geq 0$ (for maximization problems) is met. In some cases, with appropriate choice of variable to be eliminated from equality type of constraints, the above condition can be satisfied and thus dual simplex method can be applied. It is necessary to mention that dual simplex method does not require the need of artificial variables in the problem set to be solved by this method.

Symmetric Method: In case of a few methods, if one has equality type of constraints, there is no particular way to handle them except one has to convert them into inequality type of constraints. Thus, one has as many additional constraints as there are equality type constraints.

e.g. $4X_1 + 3X_2 = 5$ will be converted to:

$$4X_1 + 3X_2 \leq 5$$

$$4X_1 + 3X_2 \geq 5$$

This situation can be easily avoided by applying this technique. One example of such methods is Symmetric Method. Thus, one will not have only less rows but less columns, too, by eliminating some variables for these methods.

Similarly, more applications can be explored and applied to many problems. And much computational work can be avoided depending upon the nature of problem with respect to variables and constraints.

Chapter III

Redundancy and Inconsistency

The technique, I have discussed in previous chapters, can be further applied to any type of constraints whether it is a Type I or Type II inequality one. But, before going into that, I will discuss in this chapter other situations which should be visualized before applying this technique to inequalities, only for the sake of simplicity.

A few methods will be discussed here which involve insignificant computational work but their importance is significant. I will explore how one can identify some variables which will not appear in the solution or will make the objective function an unbounded one. Further, techniques will be discussed by which one can ignore some constraints without affecting the optimality conditions of the problem. Techniques which can detect infeasible problems prior to going to any solution methods will be also discussed.

Section 1

Extraneous Variable: By an extraneous variable, I mean a variable which will not appear in the final solution of the problem. Such variables can be identified before going to solution methods.

Method:

I. If there is any variable with a negative coefficient in the objective function of maximization problem ($X_j \geq 0$), it can be removed from the problem (i.e. objective function and constraints) without affecting the optimality of solution if

a. it has negative coefficient or missing from all the Type II constraints

b. it has positive coefficient or missing from all the Type I constraints

c. it is missing from all the equality type constraints.

In case, it is not missing, I will suggest to handle them by the technique discussed in the previous chapter and apply method described in a. and/or b. as the case may be.

Section 2

A similar method applies when there is a variable with positive coefficient in the minimization problem, and when variable is missing from the objective function ($X_j \geq 0$).

It is necessary to comment that this is an obvious result so it does not require any proof. This method can be illustrated by examples.

Ex. 3.1

Max. $X_o = 3X_1 + 3X_2 - 4X_3$

$$2X_1 + 3X_2 + X_3 \leq 12 \qquad (3.11)$$

$$6X_1 + 2X_2 - 3X_3 \geq 2 \qquad (3.12)$$

$$-X_1 - X_2 + X_3 = 1 \qquad (3.13)$$

$$X_1, X_2, X_3 \geq 0$$

From constraint 3.13, only X_3 can be eliminated to avoid complications. (One can eliminate either X_1 or X_2 but in these cases, one would have Type II constraints after elimination to ensure non-negativity on them which requires more slack variables and therefore a more complex problem.)

$$X_3 = 1 + X_1 + X_2 = 1 - (-X_1 - X_2)$$

After substitution,

$$\text{Max.} \quad X_o = -4 - X_1 - X_2$$

$$\text{S/T} \quad 3X_1 + 4X_2 \leq 11 \qquad (3.111)$$

$$3X_1 - X_2 \geq 5 \qquad (3.121)$$

$$-X_1 - X_2 \leq 1 \qquad (3.131)$$

$$X_1,\ X_2 \geq 0$$

Constraint 3.131 can be removed from the problem without losing the originality of the problem as I will discuss later on in Ex. 3.4.

Now, X_2 has negative coefficient in the objective function, positive coefficient in the Type I (3.111) and negative coef in the Type II (3.121). Thus, it can be eliminated from the problem i.e. $X_2 = 0$. The remodified problem is,

$$\text{Max.} \quad X_o = -4 - X_1$$

$$X_1 \leq \frac{11}{3} \qquad (3.112)$$

$$X_1 \geq \frac{5}{3} \qquad (3.122)$$

$$X_1 \geq 0$$

Thus, optimal solution by inspection is,

$$X_1 = \frac{5}{3} \qquad X_3 = 1 + \frac{5}{3} = \frac{8}{3}$$

$$X_o = -4 - \frac{5}{3} = -\frac{17}{3}$$

The same results were obtained by the simplex method after two iterations.

Ex. 3.2 Taking the modified form of Ex. 1.5 from Ex. 2.5,

$$\text{Max.} \quad X_o = 28 - 2X_1 - 11X_2$$

$$\text{S/T} \quad X_1 + X_2 \leq 4 \qquad (3.21)$$

$$X_1 + 2X_2 \leq 7 \qquad (3.22)$$

$$X_1 - 3X_2 \geq 2 \qquad (3.23)$$

$$X_1, X_2 \geq 0$$

X_2 has a negative coefficient in the objective function, positive coefficients in the Type I (3.21 and 3.22) and negative coefficient in the Type II (3.23). Thus, it can be removed from the problem, i.e. $X_2 = 0$. The remodified problem is,

$$\text{Max.} \quad X_o = 28 - 2X_1$$

$$X_1 \leq 4 \qquad (3.211)$$

$$X_1 \leq 7 \qquad (3.221)$$

$$X_1 \geq 2 \qquad (3.231)$$

$$X_1 \geq 0$$

Optimal solution is,

$$X_1 = 2 \qquad X_o = 28 - 4 = 24$$

$$X_3 = 7 - X_1 - 2X_2 = 7 - 2 = 5$$

The same results were obtained for the original and modified form of problems.

Unbounded Problems:

I. If any variable has a positive coefficient in the objective function of a maximization problem, then that variable and thus objective function can be increased indefiniely (or problem is unbounded) if,

a. it has negative coefficient or missing from all the Type I constraints

b. it has positive coefficient or missing from all the Type II constraints

c. it is missing from all the equality type constraints.

In case it is not missing, I will suggest to handle them by the technique discussed in the previous chapter and then, apply method described in a. and/or b. as the case may be.

II. The same method applies if any variable has a negative coefficient in the objective function of a minimization problem.

It will be illustrated by an example.

Ex. 3.3

Max. $3X_1 + 2X_2$

$$2X_1 + 3X_2 \geq 6$$

$$-X_1 + 6X_2 \leq 8$$

$$X_1, X_2 \geq 0$$

It looks obvious that X_1 can assume any value without violating the constrations and problem is unbounded.

Once again, it is necessary to mention that this is an obvious result so it does not require any proof.

The above example also verifies this statement.

However, I will caution that problems which do not have a situation as discussed, can still be unbounded. This is only a preliminary check. In some problems, this detection is not possible. There are other methods for some particular problems which will be discussed later on.

Constraints - Redundant or Infeasible Redundant Constraints:

There is at least one type of constraint which is always redundant and can always be removed simply by inspection.

If one has a constraint of the type, $-\Sigma A_i X_i \leq b_i$, $A_i \geq 0$, $b_i \geq 0$, $X_i \geq 0$, it can always be removed from the problems without affecting their optimality conditions. The reason being very obvious that negative of any positive quantity is always less than or equal to zero.

Ex. 3.4

In the example 3.1, I ignored the constraint $- X_1 - X_2 \leq 1$. It should now be clear why this was done. If this procedure is not followed, it is not possible to eliminate X_2 from the problem in Ex. 3.1.

Ex. 3.5

$$\text{Max.} \quad X_o = 5X_1 + 6X_2 - 7X_3$$

$$\text{S/T} \quad 2X_1 + X_2 + 2X_2 \leq 6 \qquad (3.51)$$

$$2X_1 + 3X_2 + X_3 \leq 12 \qquad (3.52)$$

$$-3X_1 - 4X_2 + 5X_3 \leq 6 \qquad (3.53)$$

$$X_1,\ X_2,\ X_3 \geq 0$$

X_3 has a negative coefficient in the objective function and has a positive coefficient in the Type I (3.51, 3.52 and 3.53). Thus X_3 can be removed and hence, $X_3 = 0$. The problem reduces to:

$$\text{Max.} \quad X_o = 5X_1 + 6X_2$$

$$2X_1 + X_2 \leq 6 \qquad (3.511)$$

$$2X_1 + 3X_2 \leq 12 \qquad (3.521)$$

$$-3X_1 - 4X_2 \leq 6 \qquad (3.531)$$

Now the constraint 3.531 can be removed as I have discussed. So, the problem reduces to a two variable and two constraint problem from a three variable and three constraint problem. Results were obtained the same in both the cases.

Optimal Solution: $X_1 = \frac{3}{2}$ $\quad X_2 = 3$ $\quad X_3 = 0$ $\quad X_o = \frac{51}{2}$

Infeasible Constraints: If one has the constraints of the type,

$$-A_i\ X_i = b_i$$

$$-A_i\ X_i \geq b_i$$

$A_i^j \geqslant 0 \qquad X_i \geqslant 0 \qquad b_i \geqslant 0$

The solution is always trivial. If $b_i = 0$, only one solution is possible, i.e. $X_j = 0$. If $b_i > 0$, constraints are infeasible and no solution exists.

It is quite impractical to have these types of constraints in usual practice. But, one may encounter this situation for some problems after applying the method discussed in the previous chapter.

Ex. 3.6

Max. $X_o = X_1 + X_2$

S/T $2X_1 - 3X_2 \geqslant 12$ (3.61)

$X_1 - X_2 = 5$ (3.62)

$X_1, X_2 \geqslant 0$

From 3.62, $X_1 = 5 + X_2 = 5 - (-X_2)$ thus, after substitution,

Max. $X_o = 5 + 2X_2$

S/T $-2X_2 \geqslant 2$ (3.611)

$-X_2 \leqslant 5$ (3.621)

The constraint 3.611 is infeasible and thus, no solution to this problem.

Section III

Redundancy and Inconsistency: Redundancy relates to an excess. It can be made clear by taking a very simple example. Say, a problem has two constraints, $X_1 \geqslant 3$, $X_1 \geqslant 5$. Then, we will say that the $X_1 \geqslant 3$ constraint is redundant and can be dropped from the problem. The reason being very obvious that once $X_1 \geqslant 5$ constraint is satisfied, $X_1 \geqslant 3$ constraint is automatically taken care of (as $5 \geqslant 3$).

Before specifying a few rules, the redundancy will be discussed in detail with reference to example 2.2. The constraints of this example were,

$$2X_1 + X_2 \leqslant 5 \qquad \text{I}$$

$$X_1 + 2X_2 \leqslant 12 \qquad \text{II}$$

$$3X_1 + X_2 = 5 \qquad \text{III}$$

$$X_1 + X_2 \geqslant 2 \qquad \text{IV}$$

$$10X_1 + 3X_2 \geqslant 5 \qquad \text{V}$$

Consider the first and second inequalities (both Type I). All non-negative values of X_1 and X_2 that satisfy the first inequality also satisfy the second. For example $X_1 = \frac{5}{2}$, $X_2 = 0$ and $X_1 = 0$, $X_2 = 5$ the limiting values from first constraint satisfy the second. But, the reverse is not true, i.e. $X_1 = 12$, $X_2 = 0$ and $X_1 = 0$, $X_2 = 6$ the limiting values of the second constraint satisfy the second, but do not satisfy the first constraint. So, the second constraint is redundant.

A similar situation exists between fourth and fifth constraints (both Type II). Here, $X_1 = 2$, $X_2 = 0$ and $X_1 = 0$, $X_2 = 2$ values of constraint fourth satisfy the fifth one too. But, $X_1 = 0.5$, $X_2 = 0$ and $X_1 = 0$, $X_2 = \frac{5}{3}$ satisfy the fifth constraint but not the fourth one.

So, the fifth constraint is redundant.

Now, second and fifth constraints can be dropped from the problem. Thus, the problem of Ex. 2.2 reduces to:

$$
\begin{aligned}
\text{Max.} \quad & X_o = 10X_1 + 5X_2 \\
\text{S/T} \quad & 2X_1 + X_2 \leq 5 \\
& 3X_1 + X_2 = 5 \\
& X_1 + X_2 \geq 2 \\
& X_1, X_2 \geq 0
\end{aligned}
$$

This problem gives the same solution $X_o = 25$, $X_1 = 0$, $X_2 = 5$ as was obtained with the original problem of Ex. 2.2. So, one can realize the unnecessary computations involved for Ex. 2.2 after knowing this technique and thus, importance of the technique.

To formalize the matter of redundancy with respect to inequalities, the following rules[4] can be stated:

<u>Rule 3.1</u> Given two Type I inequalities where $\frac{a_{1j}}{b_r} \leq \frac{a_{sj}}{b_s}$ for every j then r inequality is redundant with respect to the s and may be omitted from the problem. (If the above holds with the equality existing for each j the two inequalitites are identical and either is redundant.)

<u>Rule 3.2</u> Given two Type II inequalities where $\frac{a_{1j}}{b_r} \geq \frac{a_{sj}}{b_s}$ for every j then r inequality is redundant and may be omitted from the problem.

These rules hold for problems of any dimension and can be applied to pairs of inequality constraints regardless of the signs of the coefficients of the variables in the constraints and even hold if some of the coefficients equal zero.

Note, that if any variable is unrestricted in sign or is constrainted to be non-positive, then necessary changes in the coefficients of variables should be made before applying these rules, so that non-negativity restriction prevails on variables.

Inconsistency: In case, one has a mixture of constraints, i.e. equality and inequality type constraints in the problem then the following rules[4] should be applied to determine redundancy and inconsistency (infeasibility) in between equality and inequality type constraints.

Rule 3.3 Two constraints are given where $\frac{a_{rj}}{b_r} < \frac{a_{sj}}{b_s}$ for every j,

a. if the r is a Type I inequality and s is an equality, the r constraint is redundant

b. if the r is an equality and s is a Type I inequality, the problem has no feasible solution.

Rule 3.4 Two constraints are given where $\frac{a_{rj}}{b_r} > \frac{a_{sj}}{b_s}$ for every j

a. if r constraint is a Type II inequality and s constraint is an equality, the r constraint is redundant

b. if the r constraint is an equality and s is a Type II inequality, the problem has no solution.

One must note that these rules can be applied only if a non-negativity condition prevails on all the variables. Otherwise, necessary modifications must be made before one attempts to apply these rules.

The techniques discussed in this chapter have been found quite helpful in simplifying the linear programming problems. Some rules or techniques are obvious and do not require any sort of complications to implement them. These can certainly be found practicable and easy to implement to any size of the problem.

Chapter IV

Upper Bounds and Lower Bounds

In this chapter, the constraints which impose bounds on variables will be considered e.g. $X_j \geqslant 0$ impose the restriction that minimum value X_j variables can assume is zero. So, it is a type of lower bound. First, lower bound constraints will be discussed as they are easy to handle and implement.

Lower Bound Constraints: If a variable has the lower bound zero, the simplex method takes care of it. But, if there is an additional constraint $X_1 \geqslant 5$, one may add the constraint to the problem. $X_1 - S_1 + R_1 = 5$ where S_1 is slack and R_1 is artificial variable.

The other way to handle them is discussed below:

$X_1 \geqslant 5$ can be written as $X_1 - S_1 = 5$ or $X_1 = 5 + S_1 = 5-(-S_1)$. Now, one can substitute this value of X_1 in the problem as discussed in Chapter II. At the same time, one has to add the constraint $-S_1 \leqslant 5$ to ensure non-negativity on X_1.

As discussed in Chapter III, the later constraint is always redundant and thus can be omitted.

So, the modified problem will have one less constraint and two less variables for each type of such constraints. To illustrate, the following example is given.

Ex. 4.1

$$\text{Max.} \quad X_o = X_1 + 2X_2 + 3X_3$$

$$\text{S/T} \quad 2X_1 + X_2 - 4X_3 \leq 5 \qquad (4.11)$$

$$X_1 + 2X_2 + X_3 \leq 6 \qquad (4.12)$$

$$X_1 \geq 2 \qquad (4.13)$$

$$X_2, X_3 \geq 0$$

One way to handle is,

$$\text{Max.} \quad X_o = X_1 + 2X_2 + 3X_3 + 0S_1 + 0S_2 + 0S_3 - MR_3$$

$$\text{S/T} \quad 2X_1 + X_2 - 4X_3 + S_1 = 5$$

$$X_1 + 2X_2 + X_3 + S_2 = 6$$

$$X_1 - S_3 + R_3 = 2$$

Now, it can be solved by the simplex method. Another way to handle is, write $X_1 \geq 2$ as $X_1 - S_3 = 2$ or $X_1 = 2 + S_3$. After substitution,

$$\text{Max.} \quad X_o = 2 + S_3 + 2X_2 + 3X_3$$

$$\text{S/T} \quad 2S_3 + X_2 - 4X_3 \leq 1$$

$$S_3 + 2X_2 + X_3 \leq 4$$

To set up for the simplex method, it will be written as,

$$\text{Max.} \quad X_o = 2 + S_3 + 2X_2 + 3X_3 + 0S_1 + 0S_2$$

$$\text{S/T} \quad 2S_3 + X_2 - 4X_3 + S_1 = 1$$

$$S_3 + 2X_2 + X_3 + S_2 = 4$$

When the original problem was set up for simplex, it had seven variables and three constraints, while the modified problem set up for simplex had five variables and two constraints. The savings are two variables and one constraint. The solution was obtained after one iteration.

Optimal Solution: $X_o = 14 \quad S_3 = 0 \quad X_2 = 0 \quad X_3 = 4$

$$X_1 = 2 + S_3 = 2$$

The same solution was obtained with the original problem after two iterations. So, one saves the number of iterations involved. But, this may not be always true. It will be true only if the variable with lower bound constraint will appear in the final tableau with a value of more than lower bound if no lower bound was imposed on it.

Below are a few interesting results.

I. If the value of $\frac{b_i}{a_{ir}}$ for a lower bound variable X_r in any Type I or equality type constraint (all a_{ij} should be positive) is less than lower bound value, then the problem has no feasible solution.

Ex. 4.2

$$\text{Max.} \quad X_o = 2X_1 + 3X_2$$

$$\text{S/T} \quad 3X_1 + 4X_2 \leq 6 \qquad (4.21)$$

$$X_1 + X_2 = 5 \qquad (4.22)$$

$$X_1 \geq 3 \qquad (4.23)$$

$$X_2 \geq 0$$

After substitution, $X_1 = 3 + S_3$ we have,

Max. $X_o = 6 + 2S_3 + 3X_2$

S/T $\quad 3S_3 + 4X_2 \leq -3 \qquad (4.211)$

$$S_3 + X_2 = 2 \qquad (4.221)$$

the constraint 4.211 can be rewritten as $-3X_3 - 4X_2 \geq 3$ which is not feasible. So, problem has no solution.

It is necessary to note that this is an obvious result, so it does not require any proof.

II. If the positive value of $\frac{b_i}{a_{ir}}$ for a lower bound variable X_r in any Type II constraint is less than lower bound value, then this constraint will become Type I constraint after substitution.

<u>Ex. 4.3</u>

Suppose we have two constraints which are a part of some problem,

$$3X_1 + 4X_2 \geq 6 \qquad (4.31)$$

$$X_1 \geq 3 \qquad (4.32)$$

the substitution $X_1 = 3 + S_2$ will convert the Type II constraint 4.31 into Type I constraint (as $\frac{b_1}{a_{11}} = +2 < 3$)

i.e. $3S_2 + 4X_2 \geq -3$

or $\quad -3S_2 - 4X_2 \leq 3 \qquad (4.311)$

which is Type I.

The purpose of commenting on these results is just to enhance the importance of this method.

Upper Bound Constraints: The situation is not that simple with upper bound constraints. Suppose, one has the upper bound constraint $X_1 \leq 5$. After writing it in an equality form, one gets $X_1 + S_1 = 5$ or $X_1 = 5 - S_1$. So, this value of X_1 can be substituted everywhere but at the same time, one has to add the constraint $S_1 \leq 5$ to ensure non-negativity on X_1. Thus, one finds complications involved in using the technique discussed in Chapter II for upper bound constraints.

A special algorithm developed by Charnes, Lamke and Dantzig can be used to handle the problems which have some or all the variables with upper bounds. This algorithm does not require their explicit inclusion in the tableau. This method saves one constraint and one variable for each upper bound constraint.

CLD, Charnes, Lemke, Dantzig, algorith[4] is based on the following two factors:

I. UB constraints, while not explicitly included in the tableau, are never violated.

II. Variables not in the current solution are sometimes permitted to be nonzero. Only the variables having UB constraints are permitted ever to assume nonzero values while out of solution and even these can take on only one nonzero value - upper bound value, Uj. If a variable does not have upper bound constraint, it can become nonzero only by the normal method of entering the solution.

The CLD algorithm is started in the normal method as in case of standard simplex. The first solution will consist of slack and artificial variables to which we assume no upper bound constraints apply. At the first iteration X_k, the variable selected to enter the solution under the $C_j - Z_j$ rule must be structural variable.

Then:

Step 1: If X_k has no UB, it is brought into the solution under normal θ rule. According to θ rule, the variable which has to be replaced by X_k must satisfy the condition that,

$$\theta = \min \frac{b_i}{Y_{ik}} \qquad Y_{ik} > 0$$

Step 2: If X_k has an upper bound, we first compute θ.

(a) If $\theta \leq U_k$, X_k is brought into the solution in a normal iteration.

(b) If $\theta > U_k$, X_k is increased to its UB but not brought to solution. The b column is altered by forming a b' column as follows:

$$b'_i = b_i - a_{ik} U_k$$

$$i = 1, 2, \text{---}\ m \qquad 4.01$$

Make the X_k column to indicate that X_k is at its UB.

Then, re-examine the $C_j - Z_j$ row and select the next largest element, say that X_t is corresponding variable. This is treated in the same manner as X_k and the outcome may be an iteration putting X_t into the solution or we may set X_t equal to its UB. If the latter occurs, we select still another variable and repeat the process.

Step 3: Assume that now, there is an upper bounded variable in the solution at some value less then its UB. When the next iteration is performed, there is a possibility that it will be increased and exceed its UB if the element on its row of the column of the incoming variable is negative. Thus, if X_e is selected to be the incoming variable, it must be constrained to be small enough to avoid violating the UB of the other variable, say X_s. The constraint is:

$$b_s - a_{se}X_e \leq U_s$$

or

$$X_e \leq \frac{U_s - b_s}{-a_{se}} \qquad \text{if } a_{se} < 0 \qquad (4.02)$$

If X_e does not have UB, 4.02 is the only restriction on it. But, if X_e has a UB constraint, then X_e must satisfy

$$X_e = \min\left(\theta,\ U_e,\ \frac{U_i - b_i}{-a_{ie}}\right) \qquad (4.03)$$

for all constrained variables X_i now in the solution, $a_{ie} < 0$.

(a) If $X_e = 0$ under 4.03, normal iteration is performed.

(b) If $X_e = U_e$ under 4.03, X_e is increased to U_e as explained in Step 2.

(c) If $X_e = \frac{(U_i - b_i)}{-a_{ie}}$ under 4.03 for some i, say s, then X_s must be increased to its UB and replaced in the solution by X_e. This is accomplished by substituting $b_s' = b_s - U_s$ 4.04 followed by the formation of new tableau using a_{se} as the pivotal element.

Step 4: The iteration continues as per Step 3 with the following features:

(a) A variable marked as being at its UB cannot be increased at a later iteration.

(b) At each iteration, we either place those variables with the largest entry cost in the solution or increase them to their upper bounds until there are no more positive entry costs except for those variables marked as being at their UB's.

(c) If after meeting the conditions of (b), the entry cost of a variable at its UB is negative, the program can be further increased by decreasing the variable from its UB. Two precautions must be taken however:

1. If we wish to reduce X_K, say from its UB because of the condition noted, we can do it by reversing the operation of equation 4.01.

$$b_i' = b_i + a_{iK}X_K \qquad i = 1, 2, \text{--}m \qquad (4.05)$$

Suppose, however, that some variable say X_s is now in the solution but within its UB. If a_{sK} is positive, it is possible that X_s will be driven beyond its UB. We can test this for this contingency by $\Delta_k = \min_s \dfrac{U_s - b_s}{a_{sK}} \quad a_{sK} > 0 \quad (4.06.)$

For all such solutions the variable is X_s. If $\Delta_K \geqslant U_K$, we can lower X_K all the way from U_K to zero without violating the UB on any present solution variable. But if $\Delta_K < U_K$ we can only decrease X_K to $U_K - \Delta_K$ without causing some other UB to be violated. If the latter is the course we must take the following steps:

Set up the new b_i' column by 4.05. This will violate the UB constraint on X_s.

Change b_s' to $b_s' - U_s$.

Perform an iteration with a_{sk} as the pivotal element. The result will be to take X_s out of solution and replace it by X_K. Then X_s will be at its UB, U_s and X_K will be in the solution equal to $U_K - \Delta_K$

2. If the test of equation 4.06 shows that no other UB is violated when X_K is reduced all the way from U_K to zero, it is still possible that the solution will be infeasible when the new solution vector is formed by applying equation 4.05. If this happens, it means that X_K must be in the solution at some positive level to retain feasibility. The easiest way to correct the condition after (and if) the new solution vector proves to be infeasible, is to calculate θ by:

$$\theta_k' = \underset{i}{\text{Max.}} \left(\frac{b_i'}{a_{ik}}\right) \qquad \begin{matrix} b_i' < 0 \\ a_{ik} < 0 \end{matrix}$$

This rule is recommended since we know the variable that must come into the solution to correct the infeasibility.

Step 5: The entry costs are again examined for the condition of non-positive entry costs except for those variables at their UB, the latter group having non-negative entry costs in the optimal solution. If either condition is violated, additional iterations are performed under the rules already given until both are satisfied. The solution will then be optimal and satisfy all UB constraints.

It is possible for the problem to have no feasible solutions even if the problem would have feasible solutions without secondary restrictions. If there are no feasible solutions to the complete problem, there will still be artificial variables in the solution at the time the entry costs meet the requirement for an optimal solution.

Ex. 4.4 This algorithm is very useful particularly for large problems. I will illustrate it by a simple example.

$$\text{Max. } X_o = 2X_1 + 3X_2 + 4X_3$$

$$\begin{aligned} \text{S/T} \quad 5X_1 + X_2 + 2X_3 &\leq 12 \\ X_1 + 2X_2 + 2X_3 &\leq 14 \\ X_2 &\leq 3 \\ X_3 &\leq 2 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

After writing it in the standard form, it can be tabulated as:

	C_j	2	3	4	0	0						
Var.		X_1	X_2	X_3	S_1	S_2	b	0	b'	0'	b''	0''
S_1	0	5	1	2	1	0	12	6	8	8	5	1✓
S_2	0	1	2	2	0	1	14	7	10	5	4	4
C_j-Z_j		2	3	4	0	0					0	
UB			V	V								

Tableau I

	C_j	2	3	4	0	0	
Var.		X_1	X_2	X_3	S_1	S_2	b''
X_1	2	1	1/5	2/5	1/5	0	1
S_2	0	0	9/5	8/5	-1/5	1	3
C_j-Z_j		0	7/5	16/5	-2/5	0	2
UB			V	V			

Tableau II

After the first tableau is set up, we find $C_3 - Z_3$ is maximum on its row. Since $0 = 6$ (min.) exceeds $U_3 = 2$, X_3 should be increased to its UB. The b column is converted to b' in accordance with 4.01 (i.e. $b' = b - a_{i3}U3$) Since, $C_2 - Z_j$ is the next largest element in the row, we study X_2 next. When θ' is computed from b', it is found that $\theta' = 5$ (min.) which is larger than $U_2 = 3$. Therefore, X_2 is increased to UB and a second b'' vector is created by the equation 4.01 (i.e. $b'' = b' - a_{i2}U_2$).

In the row labeled UB, both X_2 and X_3 are marked as being at their UB. Now we treat X_1 as $C_1 - Z_1$ which is the third largest element on this row. Because, there is no UB on X_1, normal iteration is performed using the b'' column. X_1 drives out S_1.

Though, the maximum value read from Rableau II is 2, it actually is 19 (= 2x1 + 3x3 + 4x2).

Thus, we obtained the solution just in one iteration, while the same solution was obtained by the original problem after three iterations. At the same time, it is important to note the savings in rows and columns of the simplex tableau. This method is very useful and has been found practicable.

Chapter V

Inequality Type Constraints

After one has eliminated the extraneous variables and redundant constraints and has performed a preliminary check on the problem to determine if it is unbound or infeasible, then, he may set up the problem for the simplex method.

Up to this point, the work done has involved the equality type constraints. It is now possible to deal with the inequality type of constraints. Type I inequality type constraints are the easiest to deal with. In the case of Type II inequality constraints, one has to subtract a slack variable and add one artificial variable to ensure an initial basic feasible solution. Many methods have been suggested to deal with these constraints to reduce the number of variables. Some of them are discussed here, too. I will extend the method as applied to equality type of constraints in Chapter II, to Type II constraints.

Any Type II constraint can be converted to a Type I constraint by the method discussed in Chapter II. For instance, suppose, we have the constraint $\sum A_i X_i \geq b$, then,

$$X_r = \frac{b}{A_r} + \frac{S}{A_r} - \sum_{i=1}^{r-1} \frac{A_i}{A_r} X_i$$

$$= \frac{b}{A_r} - \left(-\frac{S}{A_r} + \sum_{i=1}^{r-1} \frac{A_i}{A_r} X_i\right)$$

Note that $A_r > 0$.

Substitute the value of X_r in the problem everywhere. At the same time, add the constraint $\sum_{i=1}^{r-1} A_i X_i - S \leq b$ to ensure non-negativity on X_r.

To illustrate its applications to different problems under different situations, I will consider several problems.

Ex. 5.1

$$\text{Max.} \quad X_o = -2X_1 - X_2$$

$$\text{S/T} \quad X_1 + 2X_2 \geq 6 \qquad (5.11)$$

$$2X_1 + 3X_2 \geq 8 \qquad (5.12)$$

$$X_1, X_2 \geq 0$$

Whenever a choice has to be made between constraints of Type II, I suggest, only for simplicity sake to look for the r constraint and variable j for which, $\frac{b_r}{a_{rj}} \geq \frac{b_s}{a_{sj}}$.

This will not only convert the r constraint to Type I but also, the s constraint will change from Type II to Type I inequality.

With this rule in mind, either X_1 or X_2 from the constraint 5.11 can be chosen. Choosing X_1, $X_1 = 6-2X_2 + S_1 = 6 - (2X_2 - S_1)$. After Substitution,

$$\text{Max.} \quad X_o = -12 + 3X_2 - 2S_1$$

$$\text{S/T} \quad 2X_2 - S_1 \leq 6 \qquad (5.111)$$

$$X_2 - 2S_1 \leq 4 \qquad (5.121)$$

Note that the constraint 5.111 ensures non-negativity on X_1.

The solution is obtained by the simplex method by one iteration and the problem has two less variables as compared to the original problem. The original problem was solved after two iterations.

Optimal Solution $S_1 = 0 \qquad X_2 = 3 \qquad X_o = -3$

$$X_1 = 6 - 6 = 0$$

Ex. 5.2

$$\text{Max.} \quad X_o = 2X_1 + X_2 + 3X_3$$

$$\text{S/T} \quad X_1 + 2X_2 + X_3 \leq 4 \qquad (5.21)$$

$$3X_1 - X_2 + 2X_3 \leq 6 \qquad (5.22)$$

$$X_1 + 8X_2 - X_3 \geq 8 \qquad (5.23)$$

$$X_1, X_2, X_3 \geq 0$$

Now the objective is to convert the constraint 5.23 from Type II to Type I. Though, any variable can be chosen for elimination. But keeping in mind the search for the appropriate variable so that the other Type I constraints are not affected by their form, I will suggest only for simplicity sake to look for the variable j such that

$$\frac{b_r}{a_{rj}} \leq \frac{b_s}{a_{sj}}$$

where the s constraint is Type I and the r constraint is a Type II inequality.

Applying this rule, X_2 should be eliminated. From the constraint 5.23,

$$X_2 = 1 + \frac{X_3}{8} + \frac{S_4}{8} - \frac{X_1}{8}$$

$$= 1 - \left(- \frac{X_3}{8} - \frac{S_4}{8} + \frac{X_1}{8}\right)$$

After substitution of the value of X_2, we get,

$$\text{Max.} \quad X_o = 1 + \frac{15}{8} X_1 + \frac{1}{8} S_4 + \frac{25}{8} X_3$$

$$\text{S/T} \quad 3X_1 + S_4 + 5X_3 \leq 8 \qquad (5.211)$$

$$25X_1 - S_4 + 15X_3 \leq 56 \qquad (5.221)$$

$$X_1 - S_4 - X_3 \leq 8 \qquad (5.231)$$

the constraint 5.231 ensures non-negativity on X_2.

The solution was obtained by the simplex method after one iteration. The original problem had one more variable and was solved after two iterations.

<u>Optimal Solution</u>: $X_o = 6 \qquad X_3 = \frac{8}{5} \qquad S_4 = 0 \qquad X_1 = 0$

$$X_2 = 1 + \frac{1}{8} \times \frac{8}{5} = \frac{6}{5}$$

Note that this method is applied only when a non-negativity condition on all variables is present. If this condition is not met, the problem should be modified such that the non-negativity condition appears before applying this method. Be advised that the number of iterations will not necessarily be reduced by applying this method. Although, most of the times, the number of iterations is reduced.

A Particular Case of Type II Constraints

Now, I will discuss a very particular case of Type II constraints. When a Type II constraint contains only one positive coefficient (all others are negative), this method becomes more interesting and effective. In this case, one will reduce two variables for each constraint of this type. Moreover, it is not necessary to add any new constraint to the problem as was the case previously. Also, necessarily, the number of iterations will be reduced. For example, assume we have the constraint,

$$-\sum_{i=1}^{r-1} A_i X_i + A_r X_r \geq b \qquad A_i \geq 0$$

then,

$$X_r = b + S + \sum_{i=1}^{r-1} A_i X_i$$

$$= b - (-S - \sum_{i=1}^{r-1} A_i X_i)$$

The value of X_r has to be substituted in the problem and at the same time, the following constraint should be added to ensure non-negativity on X_r:

i.e. $$-S - \sum_{i=1}^{r-1} A_i X_i \leq b$$

which will be redundant most of the times. Thus, effectiveness of this method is visualized.

Ex. 5.3

Max. $X_o = -3X_1 - 2X_2$

S/T $2X_1 + X_2 \geq 5$ (5.31)

$-X_1 + 2X_2 \geq 4$ (5.32)

$X_1, X_2 \geq 0$

From 5.32 constraint,

$$X_2 = 2 + \frac{1}{2} X_1 + \frac{1}{2} S_2 \qquad (5.321)$$

After substitution,

$$\text{Max. } X_o = -4 - 4X_1 - S_2$$

$$\text{S/T } \frac{5}{2} X_1 + \frac{1}{2} S_2 \geq 3 \qquad (5.311)$$

But, the constraint $\frac{-1}{2} X_1 - \frac{1}{2} S_2 \leq 2$ has to be added to ensure non-negativity on X_2. This is redundant and thus can be dropped without affecting the optimality conditions of the problem.

The modified problem has only one constraint, so it can be solved simply by inspection.

Optimal Solution: $X_1 = \frac{6}{5}$ $\qquad S_2 = 0$

$$X_o = -4 - 4 \times \frac{6}{5} = -\frac{44}{5}$$

$$X_2 = 2 + \frac{1}{2} \times \frac{6}{5} = \frac{13}{5}$$

Ex. 5.4

I will consider the problem which has two constraints of this type.

$$\text{Max. } X_o = 2X_1 + 3X_2 + 5X_3$$

$$\text{S/T } X_1 + 2X_2 + X_3 \leq 10 \qquad (5.41)$$

$$-X_1 - 2X_2 + 2X_3 \geq 4 \qquad (5.42)$$

$$2X_1 - 3X_2 - X_3 \geq 0 \qquad (5.43)$$

$$X_1, X_2, X_3 \geq 0$$

From 5.42,

$$X_3 = 2 + \frac{1}{2} X_1 + X_2 + \frac{1}{2} S_2 \qquad (5.421)$$

From 5.43,

$$X_1 = \frac{1}{2} X_3 + \frac{3}{2} X_2 + \frac{1}{2} S_3 \qquad (5.431)$$

Solving 5.421 and 5.431,

$$X_3 = \frac{8}{3} + \frac{7}{3} X_2 + \frac{1}{3} S_3 + \frac{2}{3} S_2 \qquad (5.422)$$

$$X_1 = \frac{4}{3} + \frac{8}{3} X_2 + \frac{1}{3} S_2 + \frac{2}{3} S_3 \qquad (5.432)$$

After substitution of the values of X_1 and X_3,

Max. $\quad X_o = 16 + 20\, X_2 + 4\, S_2 + 3\, S_3$

S/T $\quad 7\, X_2 + S_3 + S_2 \leq 6 \qquad (5.411)$

The constraints,

$$-7X_2 - S_3 - 2S_2 \leq 8$$

$$-8X_2 - S_2 - 2S_3 \leq 4$$

which have to be added to ensure non-negativity on X_3 and X_1, respectively are redundant.

Now the problem has only one constraint, so this problem can be solved by inspection. Optimal solution,

$S_2 = 6 \qquad X_2 = 0 \qquad S_3 = 0 \qquad X_o = 40$

$$X_1 = \frac{4}{3} + \frac{1}{3} X_6 = \frac{10}{3}$$

$$X_3 = \frac{8}{3} + \frac{2}{3} X_6 = \frac{20}{3}$$

The original problem with eight variables and three constraints was solved after four iterations.

Ex. 5.5

$$\text{Max.} \quad X_o = -X_1 - 2X_2 - X_3$$

$$\text{S/T} \quad -X_1 - X_2 + X_3 \geq 5 \qquad (5.51)$$

$$4X_1 + X_2 - X_3 \geq 1 \qquad (5.52)$$

$$X_1 - X_2 + X_3 \geq 5 \qquad (5.53)$$

$$X_1, X_2, X_3 \geq 0$$

From 5.51 constraint,

$$X_3 = 5 + X_1 + X_2 + S_1 \qquad (5.511)$$

After substitution,

$$\text{Max.} \quad X_o = -5 - 2X_1 - 3X_2 - S_1$$

$$\text{S/T} \quad 3X_1 - S_1 \geq 6 \qquad (5.521)$$

$$2X_1 + S_1 \geq 0 \qquad (5.531)$$

Note that the constraint $-X_1 - X_2 - S_1 \leq 5$ which has to be added to ensure non-negativity on X_3 is redundant.

Also, realize that X_2 has a negative coefficient in the objective function of maximization problem and is missing from the constraints, so, $X_2 = 0$.

Further note, the constraint 5.531 is redundant. Thus, the problem simplifies to,

$$\text{Max.} \quad X_o = -5 - 2X_1 - S_1$$

$$\text{S/T} \quad 3X_1 - S_1 \geq 6$$

Again, the problem has only one constraint, so it can be solved by inspection.

Optimal Solution $X_1 = 2 \qquad X_2 = 0$

$X_o = -5 - 4 = -9 \qquad X_3 = 5 + 2 = 7$

If a problem has constraints only of this type, then comments about the solution can always be made merely by simple algebraic manipulations. Three different examples will be considered to comment on their results.

Ex. 5.6

Max. $X_o = X_1 + 2X_2$

S/T $X_1 - 2X_2 \geq 4$ (5.61)

$-3X_1 + 2X_2 \geq 9$ (5.62)

$X_1, X_2 \geq 0$

From the constraints 5.61 and 5.62, the equations

$X_1 - 2X_2 - S_1 = 4$ (5.611)

$-3X_1 + 2X_2 - S_2 = 9$ (5.621)

yield

$$X_1 = -\frac{13}{2} - \frac{S_1}{2} - \frac{S_2}{2}$$

$$X_2 = -\frac{21}{4} - \frac{S_2}{4} - \frac{3}{4}S_1$$

We know that to ensure non-negativity on X_1 and X_2, the following constraints must be added.

$S_1 + S_2 \leq -13$ (5.612)

$3S_1 + S_2 \leq -21$ (5.622)

Both these constraints are infeasible. So, the solution to this problem is infeasible.

<u>Ex. 5.7</u>

Max. $X_o = 3X_1 + X_2$

S/T $-3X_1 + 4X_2 \geq 9$ (5.71)

$2X_1 - X_2 \geq 4$ (5.72)

$X_1, X_2 \geq 0$

From the constraints 5.71 and 5.72, the equations,

$-3X_1 + 4X_2 - S_1 = 9$ (5.711)

$2X_1 - X_2 - S_2 = 4$ (5.721)

give

$X_1 = 5 + \frac{1}{5} S_2 + \frac{4}{5} S_1$ (5.712)

$X_2 = 6 + \frac{3}{5} S_1 + \frac{2}{5} S_2$ (5.722)

After substitution,

Max. $X_o = 21 + 3S_1 + S_2$

S/T $-\frac{1}{5} S_2 - \frac{4}{5} S_1 \leq 5$ (5.713)

$-\frac{3}{5} S_1 - \frac{2}{5} S_2 \leq 6$ (5.723)

The constraints 5.713 and 5.723 have been added to ensure non-negativity on X_1 and X_2. But, the constraints 5.713 and 5.723 are redundant.

So, S_1 and S_2 can be increased indefinitely. Hence, objective function can be increased indefinitely. Thus, problem is unbounded.

Ex. 5.8

$$\text{Max.} \quad X_o = -X_1 - X_2$$

$$\text{S/T} \quad -3X_1 + 4X_2 \geq 9 \qquad (5.81)$$

$$2X_1 - X_2 \geq 4 \qquad (5.82)$$

$$X_1, X_2 \geq 0$$

From the constraints 5.81 and 5.82, the equations,

$$-3X_1 + 4X_2 - S_1 = 9 \qquad (5.811)$$

$$2X_1 - X_2 - S_2 = 4 \qquad (5.821)$$

give,

$$X_1 = 5 + \frac{1}{5} S_2 + \frac{4}{5} S_1 \qquad (5.812)$$

$$X_2 = 6 + \frac{3}{5} S_1 + \frac{2}{5} S_2 \qquad (5.822)$$

After substitution,

$$\text{Max.} \quad X_o = -11 - \frac{7}{5} S_1 - \frac{3}{5} S_2$$

$$\text{S/T} \quad -\frac{1}{5} S_2 - \frac{4}{5} S_1 \leq 5 \qquad (5.813)$$

$$-\frac{3}{5} S_1 - \frac{2}{5} S_2 \leq 6 \qquad (5.823)$$

the constraints 5.813 and 5.823 have been added to ensure non-negativity on X_1 and X_2. But these are redundant. Thus, solution by inspection (as there are no constraints on S_1 and S_2),

$$X_o = -11 \qquad S_1 = 0 \qquad S_2 = 0 \qquad X_1 = 5 \qquad X_2 = 6$$

the same solution will be obtained by two iterations with the simplex method.

The purpose to give many examples is to demonstrate the effectiveness of the method. Another purpose is to give fair idea to the reader how this method is implemented to the problems under different situations.

Gass's Method:

Gass has presented a method of minimizing the number of artificial variables over a set of Type II inequalities.[2]

Suppose, we have constraints $AX \geq b$ 5.01. If we write it in the standard form, we get $AX - S = b$ 5.02 where $S = (S_1, S_2, \text{-----} S_n)$ is a non-negative column vector. By applying a simple transformation to the coefficients of X, S and column vector b, we can start the computation with only one artificial vector. The scheme calls for determining $\max_i b_i = b_s$ and adding the s row to the negative of every row of 5.02.

The resulting set of n equations with (n-1) distinct positive unit vectors will require on artificial vector which corresponds to the s vector. This can be demonstrated by taking an example.

Ex. 5.9

Max. $X_o = -3X_1 - 4X_2 - 5X_3$

S/T $2X_1 + 3X_2 + 4X_3 \geq 12$ (5.91)

$X_1 + X_2 + 2X_3 \geq 6$ (5.92)

$$2X_1 + X_2 + 3X_3 \geq 4 \qquad (5.93)$$

$$X_1, X_2, X_3 \geq 0$$

The constraint 5.91 has b_1 maximum. Add the constraint 5.91 to the negative of the constraints 5.92 and 5.93, after writing them in the standard form. We get,

$$2X_1 + 3X_2 + 4X_3 - S_1 = 12$$

$$X_1 + 2X_2 + 2X_3 + S_2 - S_1 = 6$$

$$2X_2 + X_3 + S_3 - S_1 = 8$$

The modified problem is,

$$\text{Max.} \quad X_o = -3X_1 - 4X_2 - 5X_3$$

$$2X_1 + 3X_2 + 4X_3 - S_1 + R_1 = 12$$

$$X_1 + 2X_2 + 2X_3 + S_2 - S_1 = 6$$

$$2X_2 + X_3 + S_3 - S_1 = 8$$

The original problem had nine variables out of which there were three artificial. While in the modified problem, the number of variables is seven out of which is only one artificial. The solution was obtained with the original problem in two iterations while the modified problem solution was obtained in one iteration.

Optimal Solution

$$X_1 = X_2 = 0 \qquad X_3 = 3 \qquad X_o = -15$$

The reader will realize that this procedure always limits the number of artificial variables required over a set of Type II constraints to one. It, therefore, tends to minimize the number of iterations necessary to drive the objective function to an optimum.

Another method, again suggested by Gass[4], that will produce similar results is applicable to problems that contain a mixture of Type II inequalities and Type I inequalities. It will be supposed that s of m constraints are of this type and other s-m constraints are Type I inequalities and hence already have (slack) solution variables.

We first estimate which of the structural variables are most likely to be in the final solution. This estimate is made from the order of the magnitude of the coefficients of the objective function, from a knowledge of the solution from which the problem arose or from experience with similar problems. Having selected s structural variables on some basis or other, we proceed to reduce the corresponding columns to the unit vector form in all rows by a direct application of the rows' operations. One of two results will occur:

1. All elements of the b column are non-negative. If so, the solution is feasible and no artificial variables are needed. The simplex procedure is then followed in the usual manner until the optimum solution is obtained.
2. If one or more elements of the b column are negative, then follow this procedure:

A. Determine the smallest b_i (algebraically) call it b_q.

B. Subtract q row from the other rows that have negative b_i and multiply the q row itself by -1.

C. All of the b_i will now be positive, but, the operation of multiplying the q row by -1, together with subtractions involved, have destroyed the unit vector that had its 1 on the q row. So add an artificial variable to this row to establish an initial feasible solution.

The following example will demonstrate it.

<u>Ex. 5.10</u>

Max. $X_o = X_1 + X_2 + 2X_3$

S/T $2X_1 + 3X_2 + X_3 \leq 6$ (5.101)

$X_1 + 2X_2 + 3X_2 \leq 8$ (5.102)

$3X_1 - X_2 + 2X_3 \geq 4$ (5.103)

Writing the constraints in the standard form, we get:

$2X_1 + 3X_2 + X_3 + S_1 = 6$ (5.1011)

$X_1 + 2X_2 + 3X_3 + S_2 = 8$ (5.1021)

$3X_1 - X_2 + 2X_3 - S_3 = 4$ (5.1031)

First, we have to choose some structural variable which is most likely to be in the final solution. It appears that X_3 is most likely to be a candidate for the final solution.

Now performing transformations on equations, we get:

$$\frac{1}{2} X_1 + \frac{7}{2} X_2 + 0X_3 + S_1 + \frac{1}{2} S_3 = 4 \qquad (5.1012)$$

$$-\frac{7}{2} X_1 + \frac{7}{2} X_2 + 0X_3 + S_2 + \frac{3}{2} S_3 = 2 \qquad (5.1022)$$

$$\frac{3}{2} X_1 - \frac{1}{2} X_2 + X_3 - \frac{1}{2} S_3 = 2 \qquad (5.1032)$$

Thus, the initial basic feasible solution is $S_1 = 4$, $S_2 = 2$, $X_3 = 2$. The same transformation was obtained with the original problem by the simplex method after two iterations.

The next transformation, by applying the simplex method, gives the optimal solution: $X_2 = \frac{4}{7}$, $X_3 = \frac{16}{7}$, $X_1 = 2$.

By this technique, a substantial amount of savings in the number of variables as well as iterations is obtained. Moreover, one avoids the artificial variables.

After knowing all these techniques, the reader is the best judge to decide which technique will suit a particular problem. Sometimes, it will be found that a combination of one or two is useful for some problems. Either way, one can use them and implement them in any problem so as to reduce the computational work involved.

Chapter VI

A Few More Tips

In this last chapter, I will elaborate on a few more points which will be found useful, but, only for particular problems under particular situations.

A. A Particular Case of Two-Variable Problem

In the case of two variable problems, if both c_1 and c_2 are positive for maximization case and in addition, one has one Type I and another Type II inequality constraints; then, Type I inequality can be treated as an equality.

Similarly, when C_1 and C_2 are positive for minimization problem and in addition, one has one Type I and another Type II inequality can be treated as an equality. After identifying the equality, one can proceed with the method discussed in Chapter II and can solve the problem without going to the simplex method.

B. Removal of Artificial Variables

One can always avoid the artificial variables from the constraints because they do not serve any purpose. At the same time, I will caution that appropriate artificial variables must be added to the objective function. I will illustrate it by an example.

$$\text{Max.} \quad X_o = -2X_1 - X_2$$

$$\text{S/T} \quad 2X_1 + X_2 \geq 6$$

$$X_1 + X_2 \geq 4$$

$$X_1, X_2 \geq 0$$

Writing this problem in standard form,

$$\text{Max.} \quad X_o = -2X_1 - X_2 - MR_1 - MR_2$$

$$2X_1 + X_2 - S_1 = 6$$

$$X_1 + X_2 - S_2 = 4$$

It will be tablulated as,

	C_j	-2	-1	0	0	
Var.		X_1	X_2	S_1	S_2	X_o
R_1	-M	2	1	-1	0	6
R_2	-M	1	1	0	-1	4
Z_j	$-C_j$	-3M+2 ↑	-2M+1	M	M	-10M

In this way one will have less columns in the tableaus and reduce the computational work involved for these columns for each iteration.

C. Dual Simplex Method

The Dual Simplex Method is a technique by which any type of constraints can be handled and no artificial variables need be added. It is very similar to the simplex method. It can be applied only if the condition $Z_j - C_j \geq 0$ is met for maximization problems. Thus, whenever this condition can be met, it is suggested that one should proceed with the Dual Simplex Method.[7]

D. Use of Dual

When one has more constraints than the number of variables, it is suggested that one should take dual of the problem and proceed with the simplex method.[4]

E. If in a maximization problem, there is only one variable with a positive coefficient and all the constraints are of Type I inequalities, the variable with the positive coefficient must appear in the final tableau. It is very obvious, and thus, a check can be made on the final solution.

The examples which have been considered in this and previous chapters, do not attempt to cover all possible and useful manipulations. Rather, it is hoped that these suggestions will lead the reader to recognize others as he obtains experience.

Conclusion

My objective throughout the thesis was to explore various methods which can eventually result in the reduction of the size of Linear Programming problems with respect to the number of variables, number of constraints and number of iterations required to solve a particular problem consisting of inequality and equality type constraints.

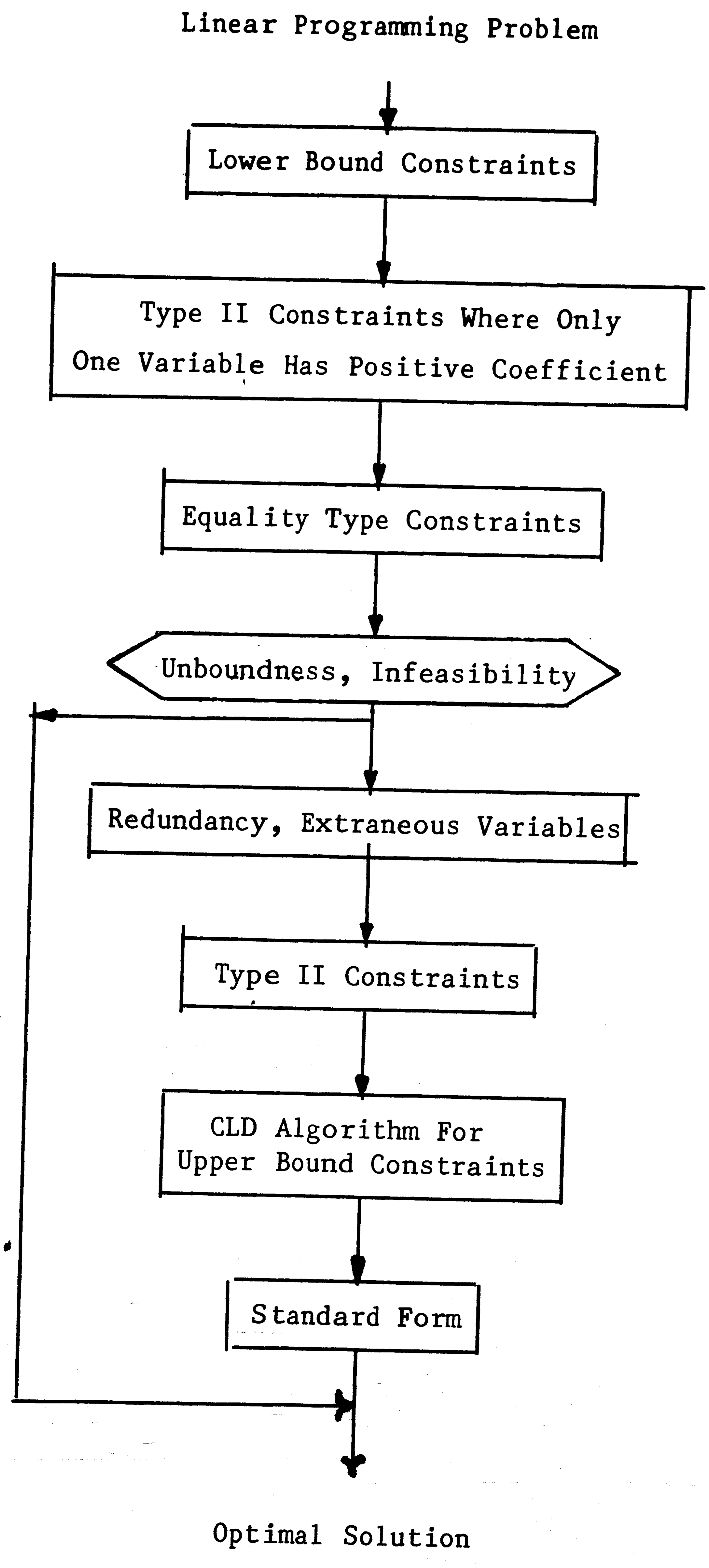
I was able to develop a technique with which the number of variables in a problem can be reduced. Discussion of this technique began by considering the equality type of constraints. Later, the work was extended to the more complex situation relating to Type II constraints. I have done a full investigation of this technique under various situations and have found it quite practical and useful.

I also have discussed techniques by which certain variables and constraints can be dropped from the problem, without affecting the optimality conditions of the problem. Further, I discussed a few situations where one can comment on infeasibility and unboundness of a problem.

I suggest a flow chart showing how one should successively apply the techniques to a Linear Programming problem as discussed below:

a) Eliminate LB constraints as discussed in Chapter 4.

b) Eliminate those Type II constraints where only one variable has positive coefficient as detailed in Chapter 5.


Linear Programming Problem
Lower Bound Constraints
Type II Constraints Where Only
One Variable Has Positive Coefficient
Equality Type Constraints
Unboundness, Infeasibility
Redundancy, Extraneous Variables
Type II Constraints
CLD Algorithm For
Upper Bound Constraints
Standard Form
Optimal Solution

c) Attack equality type constraints to reduce the number of variables as discussed in Chapter 2.

d) Make a check for unboundness and infeasibility as detailed in Chapter 3.

e) Remove redundant constraints and extraneous variables as described in Chapter 3.

f) Attack Type II constraints as discussed in Chapter 5.

g) Follow CLD algorithm for UB constraints.

h) Write the problem in the standard form.

i) Apply the Simplex method to obtain optimal solution.

It is quite possible that every problem will not have all the situations I have mentioned in the flow chart. In that case, one should bypass that item and move to the next one in sequence.

Further Areas of Study

Linear Programming problems tend to be large, resulting in lengthy calculations. Since storage space in a computer is always limited, any method that makes possible a large reduction in the size of a problem is important. So, still more investigation is useful in this field in order that efficient and effective tools can be developed to handle the large problems.

It was noted that a few constraints which in characteristic might be either Type I or Type II constraints, may still appear as equality constraints in the Final tableau. So, one can explore and investigate some definite rules which would identify such constraints.

It was found that if some of the variables could be identified which would appear in the Final tableau, it could save the arithmetic work involved in terms of the number of iterations necessary to arrive at that result. Thus, one can investigate the characteristics of variables which are likely to appear in the Final tableau.

Investigation can be extended to generalize the characteristics of unbound and infeasible problems. This will greatly help to avoid the unnecessary computational work involved in solving a problem which will give information, after many iterations, merely about infeasibility or unboundness.

Bibliography

Books

1. Saaty, Thomas L. Mathematical Methods of Operations Research. New York, New York, McGraw-Hill, 1959.

2. Gass, Saul I. Linear Programming. New York, New York, McGraw-Hill, 1964.

3. Simmonard, Michel. Linear Programming. Englewood Cliffs, New Jersey, Prentice Hall International, Inc., 1966.

4. Llewellyn, Robert E. Linear Programming. New York, New York, Holt, Rinehart, and Winston, 1964.

5. Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princeton University Press, 1963.

6. Hadley, G. Linear Programming. Reading, Massachusetts, Addison-Wesley Pub. Co., 1963.

7. Hillier, F. S. and Liberman, G. J. Introduction to Operations Research. San Francisco, Holden Day, 1967.

8. Chung, An-Min. Linear Programming. Columbus, Ohio, C. E. Merrill, 1963.

Articles

9. Thomson, G. L., Tonje, F. M., and Zionts, S. "Techniques for Removing Non-Binding Constraints and Extraneous Variables From LP Problems", Management Science, March, 1966.

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